

# Permanent current from non-commutative spin algebra

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## Abstract

We show that a spontaneous electric current is induced in a nano-scale conducting ring just by putting three ferromagnets. The current is a direct consequence of the non-commutativity of the spin algebra, and is proportional to the non-coplanarity (chirality) of the magnetization vectors. The spontaneous current gives a natural explanation to the chirality-driven anomalous Hall effect.

Persistent (permanent) current in metallic rings is an equilibrium current which can be induced when the time-reversal symmetry is broken [1]. Such a current is expected in the presence of a magnetic flux through a normal ring [2, 3] and was indeed detected experimentally [4, 5, 6]. The effect is due to a U(1) phase factor attached by the flux to the electron wave function. Here we show theoretically that a permanent current is induced in a conducting normal ring just by attaching three ferromagnets, without magnetic flux through the ring. This surprising effect can be seen in nano-scales at low temperatures. The key here is the non-commutativity of the SU(2) spin algebra, which breaks the time-reversal symmetry, and leads, in the presence of electron coherence, to a permanent electron current. Such a system would be utilized in spin polarized transport [7] or in quantum computers as a new kind of logic gates [8].

The electron has spin 1/2 (i.e., has two components), and the spin obeys SU(2) algebra. The algebra is represented by three  $2 \times 2$  Pauli matrices  $\sigma_i$  ( $i = x, y, z$ ) satisfying the commutation relation

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k, \quad (1)$$

where  $\epsilon_{ijk}$  is the totally antisymmetric tensor with  $\epsilon_{xyz} = 1$ . When a conduction electron in a conductor is scattered by some magnetic object, the electron wave function is multiplied by an amplitude  $A(\mathbf{n}) = \alpha e^{i\beta\mathbf{n}\cdot\boldsymbol{\sigma}}$ , which is generally spin-dependent and is represented by a  $2 \times 2$  matrix in spin space. Here  $\alpha$  and  $\beta$  are complex numbers and  $\mathbf{n}$  is a three-component unit vector characterizing the scattering object (such as the magnetization direction). We consider in this paper only classical, static scattering objects, and assume  $\mathbf{n}$ 's are constant vectors.

Let us consider two successive scattering events represented by  $A(\mathbf{n}_1)$  and  $A(\mathbf{n}_2)$  (Fig.1). Due to the non-commutativity of  $\sigma_i$ , the amplitude depends on the order of the scattering event;  $A(\mathbf{n}_1)A(\mathbf{n}_2) \neq A(\mathbf{n}_2)A(\mathbf{n}_1)$  in general. Various features in spin transport, which is under intensive pursuit recently [9, 10], arise from this non-commutativity. It, however, does not affect the charge transport, since the charge is given as a sum of the two spin components (denoted by  $\text{tr}$ ), and  $\text{tr}[A(\mathbf{n}_1)A(\mathbf{n}_2)] - \text{tr}[A(\mathbf{n}_2)A(\mathbf{n}_1)] = 0$ . Anomaly in the charge transport arises at the third order. We have, by virtue of Eq.1 and the relation  $\text{tr}[\sigma_i\sigma_j] = 2\delta_{ij}$ ,

$$\text{tr}[A(\mathbf{n}_1)A(\mathbf{n}_2)A(\mathbf{n}_3)] - \text{tr}[A(\mathbf{n}_3)A(\mathbf{n}_2)A(\mathbf{n}_1)] = 4\alpha^3 \sin^3 \beta \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \equiv iC_{123}, \quad (2)$$

where the cross denotes the vector product, i.e.,  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) = \sum_{ijk} \epsilon_{ijk} n_1^i n_2^j n_3^k$ . This relation

indicates that in the presence of fixed  $\mathbf{n}_i$ 's with  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \neq 0$ , the symmetry under time-reversal (more appropriately, reversal of motion) is generally broken in the charge transport. In fact, relation (2) indicates that the contribution from one path,  $x \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow x$  (Fig.1a), and its (time-) reversed one,  $x \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow x$  (Fig.1b), are not equal, and this difference results in a spontaneous electron motion in a direction specified by the sign of  $C_{123}$ , namely, a permanent current. What is essential here is the non-commutativity of the SU(2) algebra. In fact,  $C_{123}$  vanishes if all  $\mathbf{n}_i$ 's lie in a plane, in which case the algebra is reduced to a commutative U(1) algebra. The degree of the symmetry breaking,  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ , is given by the non-coplanarity, often called spin chirality (Fig.2).

The spontaneous current above would be realized on a small conducting ring with three ferromagnets with different magnetization direction,  $\mathbf{n}_1$ ,  $\mathbf{n}_2$  and  $\mathbf{n}_3$ . The ferromagnets may be attached to the ring (Fig.3a), or embedded in the ring (Fig.3b), both being within the reach of present experimental technique. In either case, the electron in a ring feels an effective spin polarization when it goes through the region ( $F_i$ ) affected by the ferromagnets, and the effect will be modeled by the exchange (spin-dependent) potential,  $V(x) = -\Delta \mathbf{n}_i \cdot \boldsymbol{\sigma}$  for  $x \in F_i$ . Here  $\Delta$  represents the effective exchange field. The equilibrium charge current in the ring is calculated from  $j(x) = \frac{\hbar e}{2m} \text{Im}(\nabla_x - \nabla_{x'}) \text{tr} G(x, x', \tau = 0-) |_{x'=x}$ , where  $G(x, x', \tau) \equiv -\langle T c(x, \tau) c^\dagger(x', 0) \rangle$  is the thermal Green function,  $e, m, c$  being the charge, the mass, the annihilation operator of electrons, respectively.  $G(x, x', \tau)$  is calculated perturbatively from the Dyson equation,  $G = g + gVG$ , where  $g$  represents free Green function. As is seen from eq. (2), possible finite current arises at the third order in  $V$ . By summing the contribution of the two paths,  $x \rightarrow X_1 \rightarrow X_2 \rightarrow X_3 \rightarrow x$  and the reversed one, we have  $j(x) = -\frac{\hbar e}{m} B(x) \text{Re } C_{123}$ . Here  $C_{123}$  is defined by eq. (2) with  $\alpha = i\Delta$ ,  $\beta = \pi/2$ , and

$$B(x) = \prod_{i=1}^3 \int_{X_i \in F_i} dX_i \int \frac{d\omega}{2\pi} f(\omega) \nabla_{X_0} \text{Im}[g_{01}g_{12}g_{23}g_{34}] |_{X_4=X_0=x} \quad (3)$$

describes the electron propagation through the ring, which is common to both paths. In Eq.3,  $f(\omega)$  is the Fermi distribution function and  $g_{ij} = g^r(X_i - X_j, \omega)$  is the retarded Green's function of free electrons. Approximating the transport along the ring as one-dimensional and neglecting multiple circulation, we have  $g^r(x, \omega) \simeq -i\pi(D/L)e^{ik_F|x|}$ , where  $k_F$  is the Fermi wavenumber,  $D$  the density of states ( $\sim 1/\epsilon_F$ ;  $\epsilon_F = \hbar^2 k_F^2 / 2m$  being Fermi energy), and  $L$  the length of the ring perimeter. The final result is given by

$$j = -2e \frac{v_F}{L} \cos(k_F L) \left( \frac{J}{\epsilon_F} \right)^3 \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3), \quad (4)$$

at zero temperature. Here  $J \equiv \pi W \Delta / L$  with  $W$  being the width of the ferromagnets, and  $v_F = \hbar k_F / m$  is the Fermi velocity.

The current is thus induced by the spin chirality  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$  of the ferromagnets. This quantity reduces to the Pontryagin index (density) for the case of smoothly varying field  $\mathbf{n}(x)$  [11], which is also interpreted as Berry phase [12] of the spin. The effect of spin Berry phase on the electron transport has so far been investigated in the limit of strong coupling to  $\mathbf{n}(x)$  where the electron spin adiabatically follows  $\mathbf{n}(x)$  [13, 14]. In contrast, the present result (4) is obtained in the opposite limit; we have treated the coupling to  $\mathbf{n}$  perturbatively (weak coupling) and made no assumption of smoothness on  $\mathbf{n}(x)$ . To see the relation between the two approaches, we follow the reasoning used in ref.[13]. We write the Hamiltonian of our system in a general form using a spatially varying polarization  $\Delta_{\mathbf{x}}$  as  $H = \sum_{\mathbf{x}} \left[ \frac{1}{2m} |\nabla c_{\mathbf{x}}|^2 + \Delta_{\mathbf{x}} \mathbf{n}_{\mathbf{x}} \cdot (c^\dagger \boldsymbol{\sigma} c)_{\mathbf{x}} \right]$  and move on to a gauge transformed frame,  $\tilde{c}_{\mathbf{x}} \equiv U_{\mathbf{x}} c_{\mathbf{x}}$ , with  $U_{\mathbf{x}} = \mathbf{M}_{\mathbf{x}} \cdot \boldsymbol{\sigma}$ . Here  $\mathbf{M}_{\mathbf{x}} = (\sin \frac{\theta}{2} \cos \phi, \sin \frac{\theta}{2} \sin \phi, \cos \frac{\theta}{2})$ , and  $(\theta, \phi)$  is the polar coordinate of  $\mathbf{n}_{\mathbf{x}}$ . The Hamiltonian is then written as  $H = \sum_{\mathbf{x}} \left[ \frac{1}{2m} |(\nabla + i\mathbf{A})\tilde{c}|^2 + \Delta_{\mathbf{x}} \tilde{c}^\dagger \sigma_z \tilde{c} \right]$ , where  $\mathbf{A} \equiv -iU^\dagger \nabla U$  is an SU(2) gauge field. In the adiabatic limit (infinitely large and uniform  $\Delta_{\mathbf{x}}$ ) on one hand, only the majority spin channel (denoted by  $+$ ) becomes relevant and hence the gauge field reduces to a U(1) field,  $\mathbf{a} \equiv \mathbf{A}_{++}$  ( $++$ -component of  $\mathbf{A}$ ). Thus the new electron,  $\tilde{c}$ , subject to a U(1) magnetic field  $(\nabla \times \mathbf{a})$ , exhibits a persistent current. In the present weakly coupled case on the other hand, both of the two spin channels in  $\tilde{c}$  are relevant and so off-diagonal components and SU(2) nature of  $\mathbf{A}$  become essential. Finite and spatially varying  $\Delta_{\mathbf{x}}$  complicates the problem further. Thus, within the adiabatic scheme, the appearance of persistent current in the perturbative regime is not obvious. Our result indicates that even in the perturbative regime, the electron feels an analog of spin Berry phase and the transport is modified.

The spontaneous current considered here is due to the memory during successive scattering events, and the electron coherence over the system is essential. The current diminishes as the temperature and/or the ring size are increased, as in the conventional persistent current due to the external magnetic flux [3]. The present current is also an equilibrium current, which cannot be measured by electrical means. At present it can only be measured by detecting its magnetic moment. The conventional persistent current was observed on a single ring of gold [5] and of GaAs-AlGaAs [6]. Compared to the conventional one, the persistent current proposed in this paper will be smaller in magnitude by a factor of  $(J/\epsilon_F)^3$ . Careful

extraction of the magnetic signal due to the persistent current from those of ferromagnets will also be required in the present case. This may be carried out by fixing one of the magnetizations, say,  $\mathbf{n}_1$ , perpendicularly to the plane containing the ring, keeping  $\mathbf{n}_2$  and  $\mathbf{n}_3$  in the plane. By changing the mutual angle between  $\mathbf{n}_2$  and  $\mathbf{n}_3$  in the plane, the spin chirality is controlled without affecting the perpendicular component of the magnetic moment. Measurement on many rings on a ferromagnetic network will be effective in amplifying the signal. (Total signal from  $N$  rings grows by a factor  $\sim \sqrt{N}$  even in the worst case where the sign of the current, represented by  $\cos k_F L$ , is random.)

The phenomenon predicted here is not restricted to artificial nano-structures, but will be present rather generally in metallic frustrated spin systems such as pyrochlore ferromagnets and spin glasses, where finite spin chirality is often realized [15, 16]. The spin chirality was recently pointed out [14], in the adiabatic limit, to be the origin of the peculiar anomalous Hall effect observed in experiments [15]. The present chirality-driven persistent current affords an intuitive interpretation to it. The circulating current starts to drift when the electric field is applied, in the direction perpendicular to the electric field (Fig.4), just as in the normal Hall effect. With the frequency of the circulating motion, read from Eq.4 as  $\Omega \simeq \frac{2\pi v_F}{L} \left(\frac{J}{\epsilon_F}\right)^3 \mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ , we may estimate the Hall conductivity by  $\sigma_{xy} = \sigma_0 \Omega \tau$  [17]. Here  $\sigma_0$  is the classical (Boltzmann) conductivity,  $\tau$  is the elastic lifetime, and the dirty case  $\Omega \tau \ll 1$  is assumed. If the spin chirality is located uniformly on every triangle of size of inter-atomic distance (i.e.,  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) = \chi_0$  and  $L \sim 1/k_F$ ), we have  $\sigma_{xy}/\sigma_0 \simeq \chi_0 J^3 \tau / \epsilon_F^2$ . This result agrees with the one obtained based on the linear response theory [18].

To summarize, we have shown that a spin chirality generally accompanies a permanent electric current. This effect is a direct consequence of the spin SU(2) algebra.

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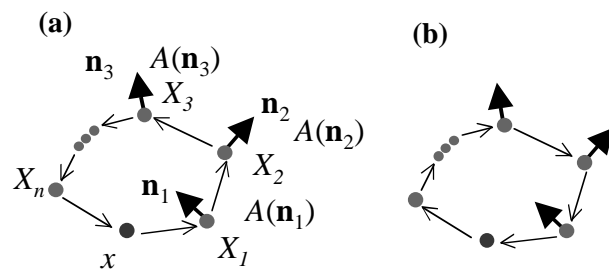
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FIG. 1: A closed path contributing to the amplitude of the electron propagation from  $x$  to  $x$ . At  $X_i$ , the electron experiences a scattering represented by an  $SU(2)$  amplitude,  $A(\mathbf{n}_i)$ . The contribution from one path (a) and the reversed one (b) are different in general due to the non-commutativity of  $A(\mathbf{n}_i)$ 's.

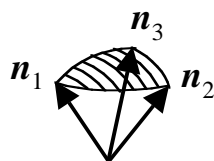
FIG. 2: Three magnetization vectors with a finite chirality,  $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3)$ .

FIG. 3: Setup for the chirality-driven permanent current. (a): Three insulating ferromagnets are put on a normal conducting ring. (b): Three metallic ferromagnets are embedded in a ring. In both cases, a conduction electron will feel the effective exchange field as it goes through the region ( $F_i$ ) affected by the ferromagnets.

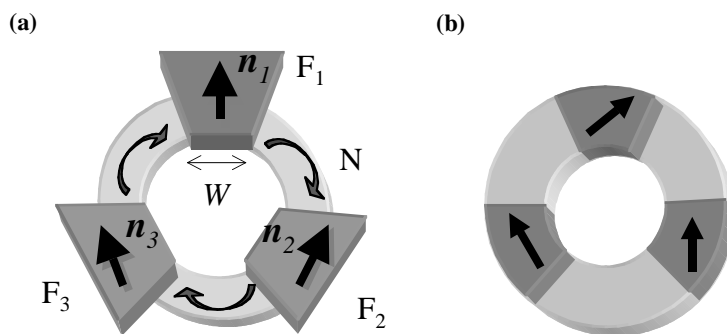
FIG. 4: Schematic picture of the chirality-induced Hall effect. Circulating permanent currents drift under applied electric field. The crosses denote impurity scattering.



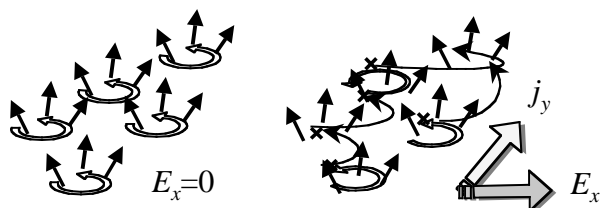
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Tatara,Kohno fig 2



Tatara,Kohno fig 3



Tatara,Kohno fig 4